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Uniqueness in linearized two-dimensional water-wave problems

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A geometrical condition sufficient for uniqueness in two-dimensional time-periodic linear water-wave problems is derived, and some examples are given. The technique can be seen as a generalization of work by John (1950), who established uniqueness for surface-piercing bodies that have the property that vertical lines down from the free surface do not intersect the body.

The paper is in two parts. Part 1 provides a review of current knowledge on the topic of uniqueness, and gives a simple form of John's proof. Part 2 describes the recent progress, in which the ideas of John's proof are extended so that uniqueness can be demonstrated for surface-piercing bodies that do not satisfy John's geometrical criterion. In fact the new technique proves uniqueness for a large class of problems involving floating bodies, submerged bodies and multiple-body systems. However, the present work still does not constitute a general proof of uniqueness for all configurations.

PART 1. A REVIEW OF PREVIOUS RESULTS

1. Statement of the problem

We consider the irrotational motion of an inviscid incompressible fluid without surface tension; we also assume that all motions are of small amplitude, and are periodic with radian frequency ω. Thus the velocity field can be expressed as the gradient of a scalar potential which can be written $\mathrm{Re}\{\Phi(x,y)\,\mathrm{e}^{-\mathrm{i}\omega t}\}$. Here (x,y) are rectangular Cartesian coordinates with origin in the mean free surface, and with the y-axis pointing vertically downwards.

The two complementary physical problems, of the radiation of waves by the forced motion of a rigid body and of the diffraction of waves by a fixed rigid body, can each be reduced to the solution of a boundary-value problem in which $\partial\Phi/\partial n$ is prescribed on the wetted surface of the body. As is usual in uniqueness problems, one considers two possible solutions Φ_1 and Φ_2, and attempts to show that the difference $\phi = \Phi_1 - \Phi_2$ vanishes. The potential ϕ must satisfy the following conditions:

$$\nabla^2\phi = 0 \quad \text{in } \mathscr{R}, \tag{1.1}$$

$$\frac{\partial\phi}{\partial y} + K\phi = 0 \quad \text{on } \mathscr{S}_{\mathrm{F}}, \tag{1.2}$$

where $K = \omega^2/g$ and g is the acceleration due to gravity;

$$\phi \sim \text{outgoing waves} \quad \text{as } x \to \pm\infty, \tag{1.3}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \mathscr{S}_{\mathrm{B}}, \tag{1.4}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \mathscr{S}_D. \tag{1.5}$$

Here $\mathscr{S}_D$ denotes the fluid bottom $y = d(x)$, where $d(x)$ is the depth of the fluid layer; in the proof that follows, it will be assumed that $d'(x)$ vanishes outside some finite range of x. (For the present, we are considering the finite-depth problem; the case of infinite depth is simpler and is briefly discussed in §2.) $\mathscr{R}$ is the region occupied by the fluid, outside any bodies present, and $\mathscr{S}_B$ is the union of the wetted surfaces of all the bodies. $\mathscr{S}_F$ is the free surface, which is the part of $y = 0$ outside all bodies. Throughout this paper a normal to a surface will always be taken *into* the fluid region.

Let $\mathscr{S}_C$ denote closures at large distance $x = \pm X$. Then Green's theorem, for the region $\mathscr{R}_S$ bounded by $\mathscr{S} = \mathscr{S}_F \cup \mathscr{S}_B \cup \mathscr{S}_C \cup \mathscr{S}_D$, gives

$$\int_{\mathscr{S}} (\phi \nabla \phi^* - \phi^* \nabla \phi)\cdot \boldsymbol{n}\, \mathrm{d}S = 0, \tag{1.6}$$

where ϕ^* denotes the complex conjugate of ϕ.

The boundary conditions (1.2), (1.4) and (1.5) ensure that the only contribution is from $\mathscr{S}_C$, and so

$$\int_{x=\pm X} \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}\right) \mathrm{d}y = 0. \tag{1.7}$$

We must now suppose that any bodies present, and any variations of $d(x)$, are in the region $x_1 \leqslant x \leqslant x_2$, where x_1 and x_2 are finite. Then it can be shown (e.g. Havelock 1929) that

$$\phi = A^{\pm} \cosh k_0(d-y)\, \mathrm{e}^{-\mathrm{i}k_0|x|} + \sum_{n=1}^{\infty} \alpha_n^{\pm} \cos \mu_n(d-y)\, \mathrm{e}^{-\mu_n |x|}$$
$$(+ \text{for } x > x_2, \ - \text{for } x < x_1). \tag{1.8}$$

In this expression k_0 is the positive root of

$$k_0 \tanh(k_0 d) = K, \tag{1.9}$$

and $\mu_1, \mu_2, \ldots$ are the ordered positive roots of

$$\mu \tan(\mu d) = -K. \tag{1.10}$$

We note here that, although the depth $d(x)$ is constant for $x > x_2$ and for $x < x_1$, these constants are not necessarily the same; the particular value is to be understood from the context. Thus k_0 and $\{\mu_n\}_{n \geqslant 1}$ may have different values in the two regions.

In (1.8) the coefficients $A^{\pm}$ are given by

$$\mathrm{e}^{-\mathrm{i}k_0|x|} A^{\pm} = \frac{2}{d}\left(1 + \frac{\sinh 2k_0 d}{2k_0 d}\right)^{-1} \int_0^d \phi(x,y) \cosh k_0(d-y)\, \mathrm{d}y$$
$$(+ \text{for } x > x_1, \ - \text{for } x < x_1). \tag{1.11}$$

We shall now show that $A^{\pm} = 0$; this is a direct consequence of the energy balance. For (1.2)–(1.5) mean that there is no energy input to the fluid region $\mathscr{R}_S$, so no energy-carrying waves can leave. Use of (1.8) in (1.7) gives

$$\int_{x=\pm X} \left(\phi \frac{\partial \phi^*}{\partial n} - \phi^* \frac{\partial \phi}{\partial n}\right) \mathrm{d}y = \tfrac{1}{4}\mathrm{i}\,|A^+|^2\,(2k_0 d + \sinh 2k_0 d)_+ + \tfrac{1}{4}\mathrm{i}\,|A^-|^2\,(2k_0 d + \sinh 2k_0 d)_-,$$

and so

$$|A^+| = |A^-| = 0. \tag{1.12}$$

Thus
$$\int_0^d \phi(x,y)\cosh k(d-y)\,\mathrm{d}y = 0 \quad \text{for } x > x_2 \text{ or } x < x_1; \tag{1.13}$$

also
$$\phi(x,y) = O(\mathrm{e}^{-\mu_1|x|}) \quad \text{as } |x|\to\infty. \tag{1.14}$$

Now consider

$$\int_{\mathscr{R}_S} |\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{R}_S} \nabla\cdot(\phi\,\nabla\phi^*)\,\mathrm{d}x\,\mathrm{d}y = -\int_{\mathscr{S}} \phi\frac{\partial\phi^*}{\partial n}\mathrm{d}s$$
$$= -\int_{\mathscr{S}_F} \phi\frac{\partial\phi^*}{\partial y}\mathrm{d}x - \int_{\mathscr{S}_C} \phi\frac{\partial\phi^*}{\partial n}\mathrm{d}y = \int_{\mathscr{S}_F} K|\phi|^2\,\mathrm{d}x - \int_{x=\pm X} \phi\frac{\partial\phi^*}{\partial n}\mathrm{d}y.$$

As $X\to\infty$ (1.14) shows that $\int_{x=\pm X}(\phi\,\partial\phi^*/\partial n)\,\mathrm{d}y\to 0$, and so

$$\int_{\mathscr{R}} |\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y = K\int_{\mathscr{S}_F} |\phi|^2\,\mathrm{d}x. \tag{1.15}$$

Furthermore, the real and imaginary parts of ϕ satisfy separately all the conditions of the problem because $A^{\pm} = 0$, and so ϕ can be taken to be real without any loss of generality. This is essential for the proof in Part 2.

2. The infinite-depth case

It is useful to set down the analogues of the results of §1, when the fluid depth is infinite, since this leads conveniently to a simple form of the uniqueness proof of John (1950). It also forms the starting point for the theory in Part 2.

When the fluid depth is infinite (1.5) is replaced by

$$|\nabla\phi|\to 0 \quad \text{as } y\to\infty. \tag{2.1}$$

For this case we take $\mathscr{S}_D$ to be a closure at large depth $y = Y$; then as $Y\to\infty$ in (1.6), (2.1) allows us to deduce (1.7) again.

Analogous to (1.8) and (1.11), we can establish that (Havelock 1929) ϕ has a representation

$$\phi = A^{\pm}\,\mathrm{e}^{-Ky-\mathrm{i}K|x|} + \int_0^\infty \alpha^{\pm}(k)\,\{k\cos(ky) - K\sin(ky)\}\,\mathrm{e}^{-k|x|}\,\mathrm{d}k, \tag{2.2}$$

where
$$\mathrm{e}^{-\mathrm{i}K|x|}A^{\pm} = 2K\int_0^\infty \phi(x,y)\,\mathrm{e}^{-Ky}\,\mathrm{d}y \quad (+\text{for } x > x_2,\ -\text{for } x < x_1). \tag{2.3}$$

Now (1.7) and (2.2) again give (1.12), so that

$$\int_0^\infty \phi(x,y)\,\mathrm{e}^{-Ky}\,\mathrm{d}y = 0 \quad \text{for } x > x_2 \text{ or } x < x_1. \tag{2.4}$$

We also require the asymptotic form of ϕ as $|x|\to\infty$; we use the fact that there are no waves to rewrite ϕ as

$$\sum_{n=1}^{\infty} \beta_n^{\pm}\,\chi_n(x,y), \tag{2.5}$$

where
$$\chi_n = \mathrm{Re}\left[\left(\frac{\mathrm{i}}{z}\right)^{n+1}(n+Kz)\right], \quad \text{with } z = y+\mathrm{i}x, \tag{2.6}$$

are wave-free potentials (see Ursell 1950; cf. also Ursell 1968). Thus

$$\phi(x,y) = O(\chi_1) = O\left(\frac{1-Ky}{x^2+y^2}\right) \quad \text{as } |x| \to \infty. \tag{2.7}$$

Again this is sufficient to show that $\int_{\mathscr{S}_C}(\phi\,\partial\phi^*/\partial n)\,\mathrm{d}y \to 0$ as $X \to \infty$, and so (1.15) applies in infinite depth also.

3. Fritz John's uniqueness proof

The proof in John (1950) is for finite depth, but is given here in its simplest form, the case of infinite depth. Further, for simplicity, we have taken the restriction to two dimensions of John's three-dimensional proof. Consider (2.4); integration by parts gives

$$\begin{aligned} 0 &= [-K^{-1}\phi(x,y)\,\mathrm{e}^{-Ky}]_0^\infty + K^{-1}\int_0^\infty \frac{\partial\phi}{\partial y}\mathrm{e}^{-Ky}\,\mathrm{d}y \\ &= K^{-1}\left\{\phi(x,0) + \int_0^\infty \frac{\partial\phi}{\partial y}\mathrm{e}^{-Ky}\,\mathrm{d}y\right\}. \end{aligned}$$

Thus

$$\phi(x,0) = -\int_0^\infty \frac{\partial\phi}{\partial y}\mathrm{e}^{-Ky}\,\mathrm{d}y. \tag{3.1}$$

Remembering that we can take ϕ to be real, we have

$$\begin{aligned} K\phi^2(x,0) &= K\left\{\int_0^\infty \frac{\partial\phi}{\partial y}\mathrm{e}^{-Ky}\,\mathrm{d}y\right\}^2 \leqslant K\left\{\int_0^\infty \left(\frac{\partial\phi}{\partial y}\right)^2\mathrm{d}y\right\}\left\{\int_0^\infty \mathrm{e}^{-2Ky}\,\mathrm{d}y\right\} \\ &= \frac{1}{2}\int_0^\infty \left(\frac{\partial\phi}{\partial y}\right)^2\mathrm{d}y \leqslant \frac{1}{2}\int_0^\infty (\nabla\phi)^2\,\mathrm{d}y. \end{aligned} \tag{3.2}$$

This applies for $x > x_2$ and $x < x_1$; we now use (1.15), which gives

$$\int_{\mathscr{R}} (\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathscr{S}_\mathrm{F}} K\phi^2(x,0)\,\mathrm{d}x \leqslant \int_{\overline{\mathscr{S}}_\mathrm{F}} K\phi^2(x,0)\,\mathrm{d}x + \frac{1}{2}\int_{\mathscr{R}_\mathrm{L}} (\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.3}$$

Here $\overline{\mathscr{S}}_\mathrm{F}$ is the part of the free surface $\mathscr{S}_\mathrm{F}$ in $x_1 \leqslant x \leqslant x_2$, and $\mathscr{R}_\mathrm{L}$ is the subset of $\mathscr{R}$ with $x > x_2$ or $x < x_1$. If the geometry allows x_1 and x_2 to be chosen such that $\overline{\mathscr{S}}_\mathrm{F}$ vanishes, then

$$\int_{\mathscr{R}} (\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{2}\int_{\mathscr{R}_\mathrm{L}} (\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{2}\int_{\mathscr{R}} (\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y, \tag{3.4}$$

so that

$$\nabla\phi \equiv 0. \tag{3.5}$$

Thus $\phi \equiv$ constant; but $\phi \to 0$ as $|x| \to \infty$, and so $\phi \equiv 0$. This proves uniqueness provided that $\overline{\mathscr{S}}_\mathrm{F}$ vanishes, which geometrically implies the following conditions.

(*a*) There must be a single body intersecting $y = 0$. Let the endpoints of the intersection be x_1 and x_2 ($x_2 > x_1$). Then

(*b*) all bodies present must lie in $x_1 \leqslant x \leqslant x_2$.

For the finite-depth case, starting from (1.13) and integrating by parts, the factor $\frac{1}{2}$ in (3.2)–(3.4) is replaced by

$$\frac{1}{2}\left(1 - \frac{2k_0 d}{\sinh 2k_0 d}\right), \tag{3.6}$$

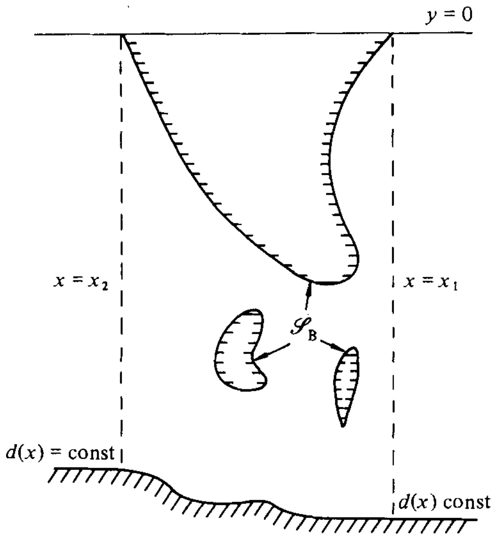

FIGURE 1. General situation where uniqueness can be proved using the method of John (1950), including depth variations and multiple bodies.

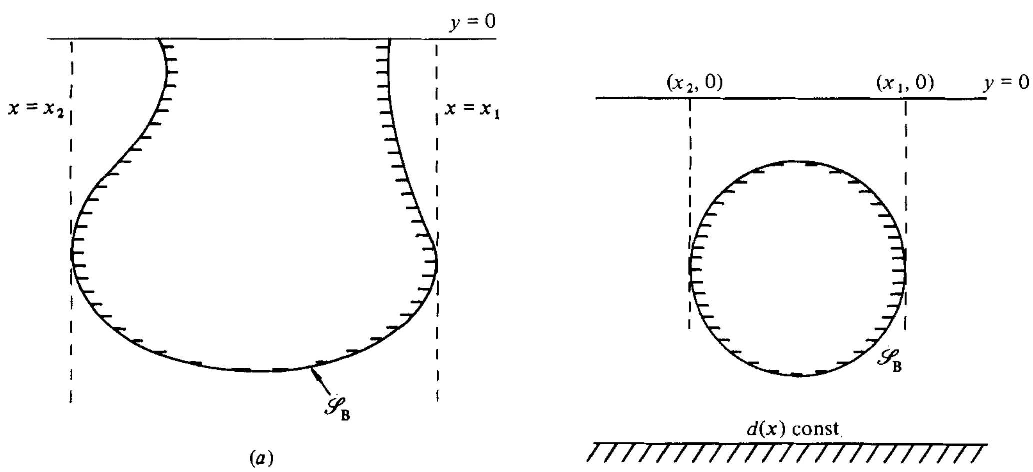

FIGURE 2. Situations where the method of John (1950) fails to prove uniqueness: (*a*) bulbous body; (*b*) totally submerged body.

and uniqueness will again follow provided $\overline{\mathscr{S}}_F$ vanishes. So the case of finite depth will be covered if a further condition is added to (*a*) and (*b*), viz

(*c*) all depth variations must lie in $x_1 \leqslant x \leqslant x_2$.

The most general situation is illustrated in figure 1. (Although John (1950) considers only the case of a single body in a uniform finite-depth ocean, the conditions necessary for uniqueness in that case leads naturally to (*a*)–(*c*) above.)

This proof fails for bulbous bodies, and for totally submerged bodies, as illustrated in figure 2. This is because the bound for $\phi^2(x,0)$ is derived from an integral on a vertical line, which must lie wholly in the fluid. This failure should not be seen as an indication of any intrinsic difficulty with the underlying hydrodynamics; the above conditions are sufficient for uniqueness, but there is no reason to believe they are necessary.

Here we should note the presence of the factor $\frac{1}{2}$ in (3.3) (or the factor (3.6) in the finite-depth case). This suggests that we might bound $\phi^2(x,0)$ by considering integrals along *non-vertical* lines, which could lead to a different (possibly larger) factor. Provided the factor is still less than unity, and every point of the free surface is used, uniqueness will follow. This approach is pursued in Part 2.

4. Other uniqueness proofs

Apart from the work of John (1950), there have been a number of papers, each of which contributes a part of our understanding of uniqueness in water-wave problems. Two of the earliest contributions were by Kreisel (1949) and Ursell (1950), both of whom made use of conformal-mapping techniques. Ursell investigated the case of the submerged circular cylinder in an infinite-depth ocean, and established uniqueness for any frequency and any depth of submergence. However, his method seems hard to extend to other geometries. Kreisel studied the effect on waves of cylindrical obstacles either in the free surface or on the fluid bottom. (This can represent waves in a fluid layer of variable depth.) He mapped the strip $-\infty < x' < \infty$, $0 < y' < d =$ constant onto the fluid domain by the transformation $z = f(z')$ and evaluated a parameter α in terms of the 'stretching' of the free surface $(\mathrm{d}z/\mathrm{d}z')_{y'=0} = h(x')$, say. Kreisel showed that, for $\alpha < 1$, the potential was defined uniquely by the asymptotic potential at one infinity. Thus, for domains that are, in some sense, close to the infinite strip of uniform depth, the diffraction and radiation problems are uniquely solvable.

Vainberg & Maz'ja (1973) also consider the wave potential in a fluid layer of variable depth, and they demonstrate two conditions under either of which the potential is unique. For the first case they suppose that the lower boundary $\mathscr{S}_{\mathrm{D}}$ is continuously differentiable, does not intersect the upper boundary ($y = 0$) and is always a finite distance from it. They show that, if

$$x n_x \leqslant 0 \quad \text{over } \mathscr{S}_{\mathrm{D}}, \tag{4.1}$$

then ϕ is unique. The second condition, which ensures uniqueness at frequency ω, is that the fluid domain should be *starlike* with respect to some point at a depth d_1, where

$$0 \leqslant d_1 \leqslant g/\omega^2. \tag{4.2}$$

These two conditions between them cover a large number of cases of variable-depth fluid layers, but it is not hard to construct situations where neither condition is satisfied, or where (4.1) does not apply and (4.2) can only be satisfied for small-enough frequency.

Condition (4.1) appears again in Fitzgerald & Grimshaw (1979), who seem unaware of Vainberg & Maz'ja's paper. They also give two more conditions for uniqueness in fluid layers of variable depth. The conditions relate to the form of the function $h(x')$, as defined above; in this respect these authors have extended Kreisel's work, and cases can be dealt with where the asymptotic depths at $x = \pm\infty$ are different.

However, the two criteria given by Fitzgerald & Grimshaw are not totally straightforward to test, since they involve the transformation (assumed known) from the uniform strip. We note here that this problem can be avoided by restating the criteria in terms of the arclength s measured along the bottom boundary of the physical domain. (This is particularly useful when the depth cannot be described by a single-valued function.) Thus, if this boundary is parametrized by $(x(s), y(s))$, then either of the following conditions will guarantee the uniqueness of the potential:

$$\left.\begin{array}{l}(a)\ y(s) \text{ is monotonic (i.e. } y'(s) \text{ is single-signed); or} \\ (b)\ y(s) \text{ is an even function and } sy'(s) \geqslant 0,\end{array}\right\} \tag{4.3}$$

for a suitable origin of s.

A further result relevant to variable-depth fluid layers is contained in Fitzgerald (1976), where some limiting cases are studied. Uniqueness is established for general bottom profiles when the frequency is sufficiently high or low; and, at a fixed frequency, the potential will be unique provided that the horizontal lengthscale of depth variations is suitably large.

The problem of a body floating in a uniform-depth ocean has been studied by Beale (1977). He showed, in a long and complicated paper using function-theoretic techniques, that the eigenvalues (that is, frequencies at which non-uniqueness could occur) form a discrete set. This work is simplified and extended by Vullierme-Ledard (1983), who shows that the eigenvalues are discrete for the infinite-depth case.† She further shows that zero frequency is not an accumulation point of the eigenvalues, and, if the body is totally submerged, then neither is infinite frequency. These conclusions about the limits as the frequency becomes small or large are confirmed with the much simpler technique in Part 2 of the present paper (see §6).

Other recent work of note (apart from that of Lenoir & Martin (1981), whose uniqueness proof has been shown to be erroneous) has concentrated on submerged bodies. Kershaw (1983) uses integral-equation methods to show that the potential will be unique provided the body is strictly convex and is deeply enough submerged. However, it seems very hard to derive from his method just how deep the body must be, even for the simplest geometry. Kershaw's result is very similar to (though not covered by) a result of Hulme (1984), referred to in §5. Hulme's work concentrates on a uniqueness criterion originally stated by V. G. Maz'ja, in Russian; the English translation of his work (in a very condensed form) appears in Maz'ja (1978). The criterion can be simply stated as follows: define $v = (x(y^2-x^2), -2x^2y)$, then the potential is unique provided

$$\boldsymbol{v}\cdot\boldsymbol{n} \geqslant 0 \quad \text{over } \mathscr{S}_{\mathrm{B}} \text{ (and } \mathscr{S}_{\mathrm{D}} \text{ if in finite depth).} \tag{4.4}$$

Although this is a simple condition, Maz'ja does not discuss what types of submerged bodies (or variable-depth fluid layers) will satisfy it. Hulme illustrates its geometrical interpretation and gives several examples of configurations where (4.4) is satisfied; figure 3 reproduces figure 2 of Hulme's paper (with his permission). This shows the integral curves of $\boldsymbol{v}$, defined by

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y};$$

they are a family of directed semicircular arcs, starting on the y-axis and terminating at the origin. Condition (4.4) is then equivalent to the statement that all such arcs point into the fluid region over the whole of $\mathscr{S}_{\mathrm{B}} \cup \mathscr{S}_{\mathrm{D}}$. This geometrical interpretation makes it much easier to determine whether or not a particular configuration will satisfy Maz'ja's criterion, and Hulme gives an example, in his figure 3, of a body for

† Beale and Vullierme-Ledard both work in three dimensions, but their work is still applicable to the two-dimensional case.

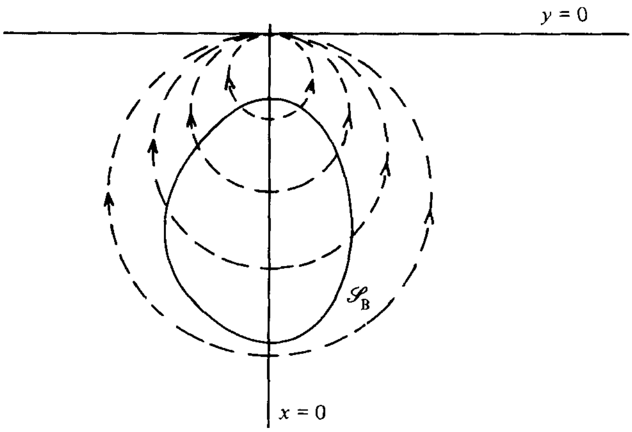

FIGURE 3. The integral curves of $\boldsymbol{v} = (x(y^2-x^2), -2x^2y)$ and the boundary of a body that satisfies Maz'ja's criterion.

which Maz'ja's uniqueness proof does not work, as can be seen immediately from the diagram.

We now move on to describe our recent progress on uniqueness.

PART 2. AN EXTENSION OF JOHN'S PROOF

5. Integrals along non-vertical lines

We start by considering infinite depth. Green's theorem gives an analogue of (2.4) for lines that are not vertical, viz

$$\int_{\mathscr{C}_b}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s = 0. \tag{5.1}$$

Here ψ represents any wave potential that is bounded as $|x|\to\infty$, and $\mathscr{C}_b$ is any line between the free surface and $y=\infty$, provided there are no bodies present in the fluid between $\mathscr{C}_b$ and $x=\infty$. Here s is arclength measured along $\mathscr{C}_b$ (see figure 4). Note that we are dealing only with positive x; bounds for $\phi^2(b,0)$ where $(b,0)$ is on the negative-x side of all floating bodies, can be dealt with separately in an identical way.

Now we make a specific choice of ψ:

$$\psi = \mathrm{e}^{-Ky-\mathrm{i}Kx}. \tag{5.2}$$

This choice gives

$$\frac{\partial\psi}{\partial x} = -\mathrm{i}K\psi = \mathrm{i}\frac{\partial\psi}{\partial y},$$

and so

$$\frac{\partial\psi}{\partial n} = \mathrm{i}\frac{\partial\psi}{\partial s}. \tag{5.3}$$

(Alternatively, this can be seen from the fact that ψ is an analytic function of $y+\mathrm{i}x$, and is thus analytic in $s+\mathrm{i}n$ when referred to the local axes $\boldsymbol{n}$, $\boldsymbol{s}$.) This last equation is the key step, since (5.1) now gives

$$\int_{\mathscr{C}_b}\psi\frac{\partial\phi}{\partial n}\mathrm{d}s = \int_{\mathscr{C}_b}\phi\frac{\partial\psi}{\partial n}\mathrm{d}s = \mathrm{i}\int_{\mathscr{C}_b}\phi\frac{\partial\psi}{\partial s}\mathrm{d}s = \mathrm{i}[\phi\psi]-\mathrm{i}\int_{\mathscr{C}_b}\psi\frac{\partial\phi}{\partial s}\mathrm{d}s,$$

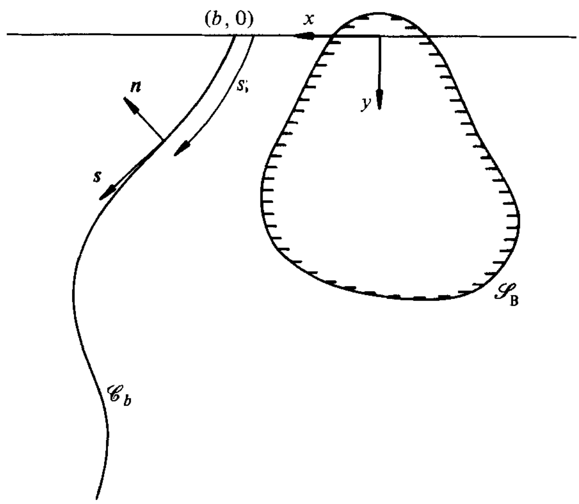

FIGURE 4. Fluid domain, axes and a line $\mathscr{C}_b$.

or
$$\phi(b,0)\,e^{-iKb} = i\int_{\mathscr{C}_b}\psi\frac{\partial\phi}{\partial n}\,ds - \int_{\mathscr{C}_b}\psi\frac{\partial\phi}{\partial s}\,ds. \tag{5.4}$$

Thus
$$\phi(b,0) = \int_{\mathscr{C}_b}\left(i\frac{\partial\phi}{\partial n}-\frac{\partial\phi}{\partial s}\right)e^{-Ky-iK(x-b)}\,ds. \tag{5.5}$$

This is the required analogue of (3.1), and we continue by bounding $\phi^2(b,0)$; again we remember that ϕ is real:

$$\begin{aligned}\phi^2(b,0) &= \left\{\int_{\mathscr{C}_b}\left(i\frac{\partial\phi}{\partial n}-\frac{\partial\phi}{\partial s}\right)e^{-Ky-iK(x-b)}\,ds\right\}^2 \\ &\leqslant \left\{\int_{\mathscr{C}_b}\left|\left(i\frac{\partial\phi}{\partial n}-\frac{\partial\phi}{\partial s}\right)e^{-Ky-iK(x-b)}\right|ds\right\}^2 \\ &= \left\{\int_{\mathscr{C}_b}|\nabla\phi|\,e^{-Ky}\,ds\right\}^2 = \left\{\int_{\mathscr{C}_b}|\nabla\phi|\,e^{-Ky}\frac{ds}{dy}\,dy\right\}^2 \\ &\leqslant \left\{\int_{\mathscr{C}_b}e^{-2Ky}\left(\frac{ds}{dy}\right)^2dy\right\}\left\{\int_{\mathscr{C}_b}|\nabla\phi|^2\,dy\right\};\end{aligned} \tag{5.6}$$

and so
$$K\phi^2(b,0) \leqslant m(\mathscr{C}_b)\left\{\int_{\mathscr{C}_b}|\nabla\phi|^2\,dy\right\}, \tag{5.7}$$

where
$$m(\mathscr{C}_b) = K\int_{\mathscr{C}_b}e^{-2Ky}\left(\frac{ds}{dy}\right)^2dy. \tag{5.8}$$

We will have achieved our aim provided that we can construct a set of lines $\mathscr{C}_b$, described by $x = x(y;b)$, such that

$$\left.\begin{array}{l}(a)\text{ the lines are non-intersecting, and } N(\mathscr{C}_b) = \inf_y(\partial x/\partial b) > 0;\\ (b)\text{ every point } (b,0) \text{ of } \mathscr{S}_F \text{ corresponds to one line } \mathscr{C}_b;\text{ and}\\ (c)\ m(\mathscr{C}_b) \leqslant MN(\mathscr{C}_b) \text{ for all lines } \mathscr{C}_b, \text{ where } M \leqslant 1.\end{array}\right\} \tag{5.9}$$

If these conditions are satisfied, then the lines are space-filling and

$$\int_{\mathscr{R}} (\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y = K\int_{\mathscr{S}_\mathrm{F}} \phi^2(b,0)\,\mathrm{d}b \leqslant \int_{\mathscr{S}_\mathrm{F}} \left[m(\mathscr{C}_b)\int_{\mathscr{C}_b}(\nabla\phi)^2\,\mathrm{d}y\right]\mathrm{d}b$$
$$\leqslant \int_{\mathscr{R}_\mathrm{L}} \frac{m(\mathscr{C}_b)}{N(\mathscr{C}_b)}(\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant M\int_{\mathscr{R}_\mathrm{L}} (\nabla\phi)^2\,\mathrm{d}x\,\mathrm{d}y, \tag{5.10}$$

where $\mathscr{R}_\mathrm{L}$ is the region swept out by all the lines $\mathscr{C}_b$. Clearly $\nabla\phi \equiv 0$ if $M < 1$, and so $\phi = \text{const} = 0$. Also, if $M = 1$, then uniqueness follows provided $\mathscr{R}_\mathrm{L}$ is not the whole of $\mathscr{R}$; for then $\nabla\phi = 0$ in the portion of $\mathscr{R}$ not covered by $\mathscr{R}_\mathrm{L}$, but ϕ is real-analytic, and so $\nabla\phi = 0$ everywhere.

It should be made clear at this stage that only two of the lines $\mathscr{C}_b$ are really significant, namely two that (together with a closure at large depth) bound the volume of fluid that contains all the bodies present, and that contains no open interval of the free surface. All other lines $\mathscr{C}_b$ can; if required, be formed by a horizontal displacement of one or other of the two bounding lines, depending only on which side of the bodies they lie. This means we can take $N(\mathscr{C}_b) = 1$ for all $\mathscr{C}_b$, and so we require, for uniqueness, simply that $m(\mathscr{C}_b) \leqslant 1$ on the two bounding lines.

Clearly this proof includes the John (1950) proof, since vertical lines have $\mathrm{d}s/\mathrm{d}y = 1$, so $m = \frac{1}{2}$; the current proof, however, also allows $\mathrm{d}s/\mathrm{d}y > 1$. To employ the proof, we must use specific functions and the possibilities are unlimited. We start with the simplest case, that of straight lines at an angle β to the vertical.
Thus

$$\frac{\mathrm{d}s}{\mathrm{d}y} = \sec\beta, \quad m(\mathscr{C}_b) = K\sec^2\beta\int_0^\infty \mathrm{e}^{-2Ky}\,\mathrm{d}y = \tfrac{1}{2}\sec^2\beta, \tag{5.11}$$

which allows

$$\sec\beta \leqslant \sqrt{2}, \quad \text{i.e. } \beta \leqslant \tfrac{1}{4}\pi. \tag{5.12}$$

Immediately we can show many problems are unique, as the examples in figure 5 illustrate.

It is interesting to compare the conclusions here with those for a submerged body under Maz'ja's theorem (Hulme 1984). Consider the submerged elliptical cylinder

$$\frac{x^2}{\lambda^2} + (y-h)^2 = a^2 \quad (h > a \text{ and } \lambda > 0). \tag{5.13}$$

The present work proves uniqueness provided that

$$\frac{h}{a} \geqslant (1+\lambda^2)^{\frac{1}{2}}, \tag{5.14a}$$

while Hulme obtains

$$\frac{h}{a} \geqslant \max(2\lambda^2 - 1, 1). \tag{5.14b}$$

Comparing these results, we see that the present work is an improvement for $\lambda > \frac{1}{2}\sqrt{5}$. Note, however, the poor result of the present work when $\lambda \leqslant 1$, where Maz'ja's theorem shows that the cylinder gives a unique potential at any depth of submergence.

Consider also the general result stated in Corollary 2 of Hulme's paper. He showed

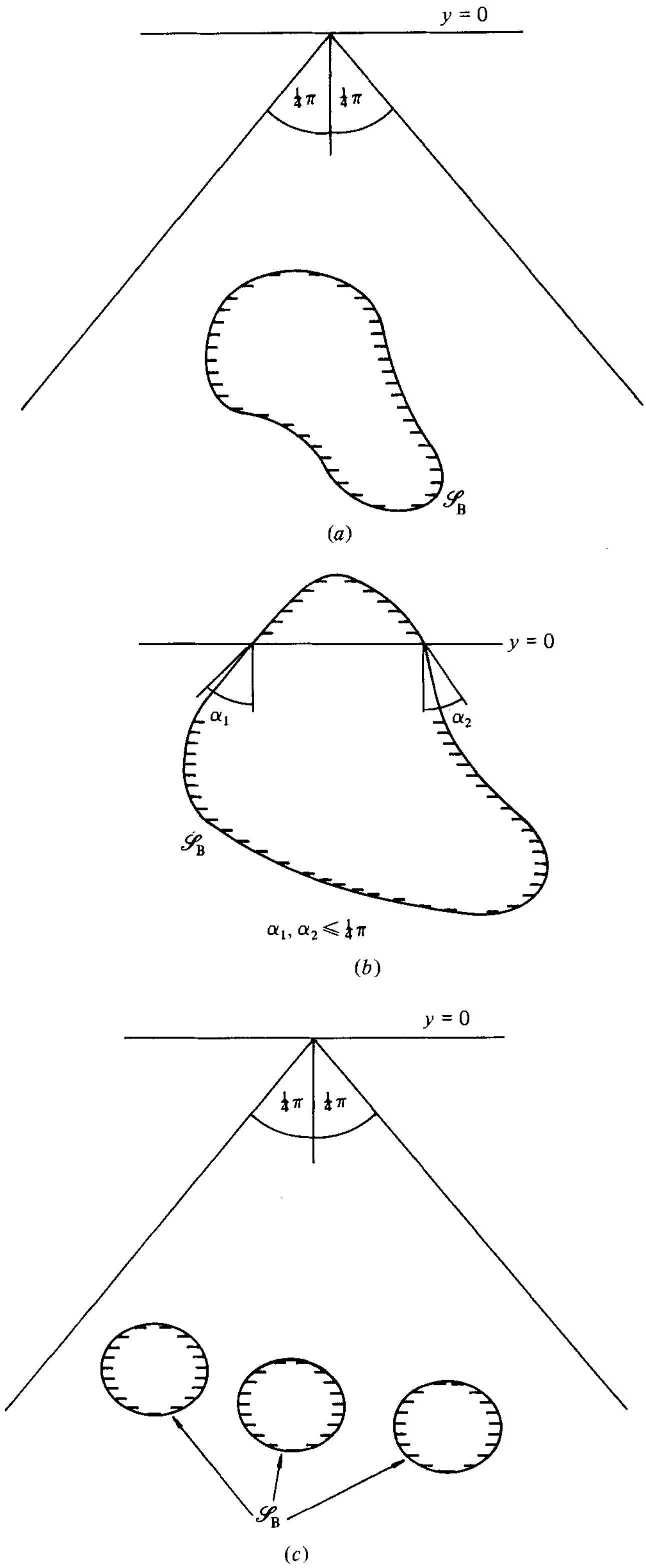

FIGURE 5. Situations shown to be unique by the present proof: (*a*) a submerged body; (*b*) a floating body; (*c*) a multiple-body problem.

that the potential ϕ will vanish if (a) $xn_x > 0$ $(x \neq 0)$ over $\mathscr{S}_B$, and (b) the body is deeply enough submerged.

Conversely, the present work shows that $\phi \equiv 0$ for any single body, or multiple-body system, provided it is (a) of finite horizontal extent, and (b) deeply enough submerged.

For a *particular* body, the previous example shows that the present work may be stronger or weaker than Maz'ja's theorem in estimating the minimum depth of submergence for uniqueness.

6. Frequency-dependent results

We move on to consider lines $\mathscr{C}_b$ that are not straight. Since $\mathrm{d}s/\mathrm{d}y$ is no longer constant, the uniqueness proof is now frequency-dependent (see later).

$$m(\mathscr{C}_b) = K\int_0^\infty \mathrm{e}^{-2Ky}\left(\frac{\mathrm{d}s}{\mathrm{d}y}\right)^2 \mathrm{d}y = \tfrac{1}{2} + K\int_0^\infty \mathrm{e}^{-2Ky}\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 \mathrm{d}y \leqslant 1,$$

so we require

$$2K\int_0^\infty \mathrm{e}^{-2Ky}\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 \mathrm{d}y \leqslant 1 \tag{6.1}$$

to satisfy (5.9c).

The case of straight lines at an angle $\beta \leqslant \frac{1}{4}\pi$ to the vertical will be referred to as Case A. Two other simple geometries are as follows:

Case B

$$\frac{\mathrm{d}|x|}{\mathrm{d}y} = \begin{cases} 0 & (0 \leqslant y \leqslant l), \\ \tan\beta & (l < y < \infty), \end{cases} \tag{6.2}$$

so that (6.1) gives

$$\tan\beta \leqslant \mathrm{e}^{Kl}; \tag{6.3}$$

Case C

$$\frac{\mathrm{d}|x|}{\mathrm{d}y} = \begin{cases} \tan\beta & (0 \leqslant y \leqslant l) \\ 0 & (l < y < \infty), \end{cases} \tag{6.4}$$

in which case

$$\tan\beta \leqslant (1-\mathrm{e}^{-2Kl})^{-\frac{1}{2}}. \tag{6.5}$$

These cases are useful because the maximum angle β obtainable is greater than $\frac{1}{4}\pi$ in each case. Examples of their use are shown in figure 6. Case B proves very useful for submerged bodies, and for high frequency. For instance, if we return to the submerged elliptical cylinder, (5.14a) can be replaced by

$$\frac{h}{a} \geqslant \frac{l}{a} + (1+\lambda^2\mathrm{e}^{-2Kl})^{\frac{1}{2}}. \tag{6.6}$$

It is clear that very small depths of submergence are possible provided that K is large. It is interesting to regard the right-hand side of (6.6) as a function of Kl (for fixed Ka) and then optimise the bound. We can show that the optimum can improve upon (5.14a) if $Ka > \lambda^{-2}(1+\lambda^2)^{\frac{1}{2}}$, giving for example

$$\frac{h}{a} \geqslant \frac{4+3\ln 2}{2\sqrt{2}} \approx 2.15 \quad \text{when } \lambda = 2\sqrt{2} \text{ and } Ka = \sqrt{2}.$$

Case B also allows the proof of uniqueness for *any* finite submerged body, provided the frequency is high enough. This is a result to be expected but unavailable from the frequency-independent proofs of John and Maz'ja. A sketch of this proof is as follows.

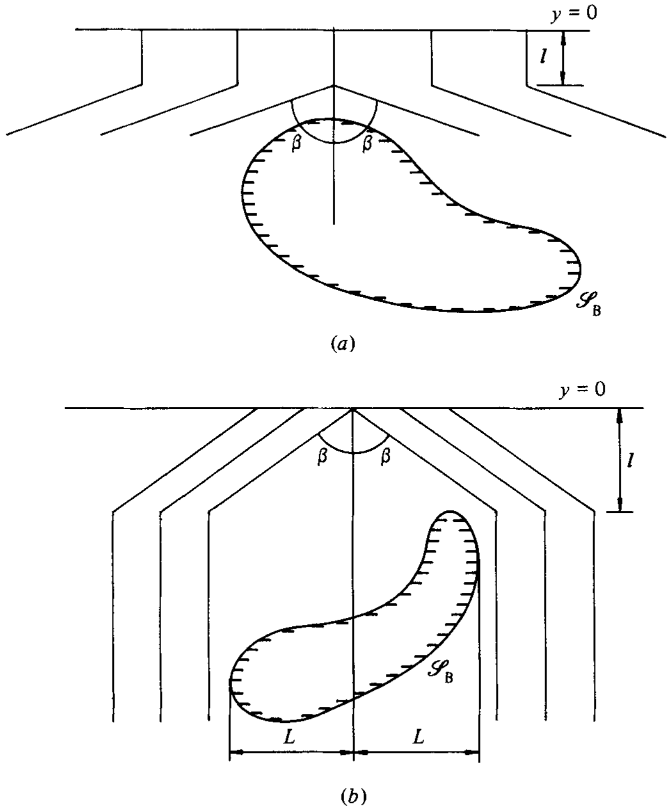

FIGURE 6. Lines $\mathscr{C}_b$ for the proof of uniqueness for the (a) high-frequency, and (b) low-frequency, limits.

Let the highest point of the submerged body be at $y = L > 0$, and choose $l \in (0, L)$. Take the set of lines $\mathscr{C}_b$ as in Case B with identical slopes β, sloping away from the highest point on either side of the body (see figure 6a). These curves are indeed non-intersecting, and each point of the free surface is used. Further, by construction there will be some value of $\beta(<\frac{1}{2}\pi)$ large enough so that the lines do not intersect the body. Then uniqueness follows provided (6.3) applies, that is

$$K \geqslant \frac{1}{l} \ln (\tan \beta). \tag{6.7}$$

The complementary Case C also proves useful, particularly when K is small. In fact, it allows the proof of uniqueness for any body of finite extent, floating or submerged,† provided the frequency is low enough. The proof runs as follows.

Choose the origin of x such that the body lies between $x = -L$ and $x = +L$. Draw two lines from the origin, sloping with equal angles β, one on each side of $x = 0$. For $l < y < \infty$ continue these lines vertically (see figure 6b). All other lines $\mathscr{C}_b$ will be identical to one or other of these two, depending on which side of $x = 0$ they lie.

From the geometry of the problem, the lines $\mathscr{C}_b$ will not intersect the body provided

$$\beta \geqslant \alpha \quad \text{for some } \alpha < \tfrac{1}{2}\pi \quad \text{and} \quad l \tan \beta \geqslant L. \tag{6.8a,b}$$

† The proof will be given here for a submerged body, but the floating case follows exactly the same ideas; the proof will not apply when a body intersects the free surface tangentially.

Now (6.5) and (6.8a) give

$$(1-\mathrm{e}^{-2Kl})^{-\frac{1}{2}} \geqslant \tan\alpha \quad \text{or} \quad Kl \leqslant -\tfrac{1}{2}\ln(1-\cot^2\alpha). \tag{6.9}†$$

But (6.8b) can be written

$$KL \leqslant \frac{Kl}{(1-\mathrm{e}^{-2Kl})^{\frac{1}{2}}} \tag{6.10}$$

and the right-hand side is an increasing function of Kl, so that (6.9) and (6.10) put an upper bound on KL. Thus uniqueness is proved for

$$K \leqslant -\frac{\tan\alpha}{2L}\ln(1-\cot^2\alpha). \tag{6.11}$$

Note that this uniqueness proof could, in principle, have been carried out with $|\mathrm{d}x/\mathrm{d}y|$ = any monotonic decreasing function of y; similarly Case B could be replaced by having any monotonic increasing function of y for $|\mathrm{d}x/\mathrm{d}y|$. Examples of these are

$$\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| = c(Ky)^n \quad (n \geqslant -\tfrac{1}{2}), \tag{6.12a}$$

and (6.1) gives

$$c^2 \leqslant 2^{2n}/\Gamma(2n+1). \tag{6.12b}$$

Notice that the curves depend on K even though the bound does not; the alternative is to have curves independent of K (e.g. $|\mathrm{d}x/\mathrm{d}y| = f(y/l)$, where l is a lengthscale of the problem), but then the bound does involve K via Kl.

The results of this section agree with the conclusions of Vullierme-Ledard, already mentioned in §4, but go further. Not only does the present work demonstrate that non-uniqueness cannot occur near zero frequency (nor near infinite frequency if the body is submerged), but it also provides bounds on K. That is, for a particular body, Cases B and C give the range of frequencies inside which any non-uniqueness will occur (if at all).

7. The finite-depth case

Put

$$\psi = \cosh k_0(d-y)\,\mathrm{e}^{-\mathrm{i}k_0(x-b)} \quad \text{and} \quad \overline{\psi} = \mathrm{i}\sinh k_0(d-y)\,\mathrm{e}^{-\mathrm{i}k_0(x-b)}, \tag{7.1}$$

where k_0 is as defined in (1.9). Both ψ and $\overline{\psi}$ are harmonic; ψ satisfies (1.2) and (1.5), whereas $\overline{\psi} = 0$ on $y = d$. Here we are assuming that $d(x) = d$ = constant between $\mathscr{C}_b$ and $x = \infty$ (cf. figure 4). Also

$$\frac{\partial\psi}{\partial x} = -\mathrm{i}k_0\psi = \frac{\partial\overline{\psi}}{\partial y} \quad \text{and} \quad \frac{\partial\psi}{\partial y} = \mathrm{i}k_0\overline{\psi} = -\frac{\partial\overline{\psi}}{\partial x}, \tag{7.2}$$

so that

$$\frac{\partial\psi}{\partial n} = \frac{\partial\overline{\psi}}{\partial s}. \tag{7.3}$$

Again

$$\int_{\mathscr{C}_b}\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}s = 0,$$

† Notice that, if $\alpha \leqslant \frac{1}{4}\pi$, Case A shows the situation to be unique at all frequencies.

where $\mathscr{C}_b$ stretches between $(b, 0)$ and $y = d$, so that

$$\int_{\mathscr{C}_b} \psi \frac{\partial \phi}{\partial n} \mathrm{d}s = \int_{\mathscr{C}_b} \phi \frac{\partial \psi}{\partial n} \mathrm{d}s = \int \phi \frac{\partial \overline{\psi}}{\partial s} \mathrm{d}s = [\phi \overline{\psi}]_{\mathscr{C}_b} - \int_{\mathscr{C}_b} \overline{\psi} \frac{\partial \phi}{\partial s} \mathrm{d}s$$

$$= -\mathrm{i}\phi(b, 0) \sinh(k_0 d) - \int_{\mathscr{C}_b} \overline{\psi} \frac{\partial \phi}{\partial s} \mathrm{d}s.$$

Hence

$$\sinh(k_0 d)\, \phi(b, 0) = \mathrm{i} \int_{\mathscr{C}_b} \left(\psi \frac{\partial \phi}{\partial n} + \overline{\psi} \frac{\partial \phi}{\partial s} \right) \mathrm{d}s$$

$$= \int_{\mathscr{C}_b} \left\{ \mathrm{i} \cosh k_0(d-y) \frac{\partial \phi}{\partial n} - \sinh k_0(d-y) \frac{\partial \phi}{\partial s} \right\} \mathrm{e}^{-\mathrm{i}k_0(x-b)} \mathrm{d}s. \quad (7.4)$$

As before, we must now bound $\phi^2(b, 0)$; in order to achieve a bound that is useful for small $k_0 d$, we take the real part. Remembering that ϕ is real, we have

$\sinh(k_0 d)\, \phi(b, 0)$

$$= \int_{\mathscr{C}_b} \left\{ \cosh k_0(d-y) \sinh k_0(x-b) \frac{\partial \phi}{\partial n} - \sinh k_0(d-y) \cos k_0(x-b) \frac{\partial \phi}{\partial s} \right\} \mathrm{d}s. \quad (7.5)$$

We use the inequality

$$\left| \cosh k_0(d-y) \sin k_0(x-b) \frac{\partial \phi}{\partial n} - \sinh k_0(d-y) \cos k_0(x-b) \frac{\partial \phi}{\partial s} \right|^2$$

$$\leqslant \{\cosh^2 k_0(d-y) \sin^2 k_0(x-b) + \sinh^2 k_0(d-y) \cos^2 k_0(x-b)\} (\nabla \phi)^2$$

$$= \{\sinh^2 k_0(d-y) + \sin^2 k_0(x-b)\} (\nabla \phi)^2. \quad (7.6)$$

Then (1.9), (7.5) and (7.6) give

$$K\phi^2(b, 0) \leqslant \frac{2k_0 d}{\sinh 2k_0 d} \left[\frac{1}{d} \int_{\mathscr{C}_b} \{\sinh^2 k_0(d-y) + \sin^2 k_0(x-b)\} \left(\frac{\mathrm{d}s}{\mathrm{d}y} \right)^2 \mathrm{d}y \right] \left[\int_{\mathscr{C}_b} (\nabla \phi)^2 \mathrm{d}y \right]. \quad (7.7)$$

Once again it will be sufficient if we can show that the expression multiplying $\int_{\mathscr{C}_b} (\nabla \phi)^2 \mathrm{d}y$ is less than unity, and again it is necessary to use a specific function $x(y)$. As before, the most convenient case is that of straight lines, even though these may not give the strongest bound.

Thus

$$\frac{1}{d} \int_{\mathscr{C}_b} \{\sinh^2 k_0(d-y) + \sin^2 k_0(x-b)\} \left(\frac{\mathrm{d}s}{\mathrm{d}y} \right)^2 \mathrm{d}y$$

$$= \frac{1}{d} \sec^2 \beta \int_0^d \{\sinh^2(k_0 y) + \sin^2(k_0 y \tan \beta)\} \mathrm{d}y$$

$$= \frac{\sec^2 \beta}{2d} \int_0^d \{\cosh(2k_0 y) - \cos(2k_0 y \tan \beta)\} \mathrm{d}y$$

$$= \tfrac{1}{2} \sec^2 \beta \left\{ \frac{\sinh(2k_0 d)}{2k_0 d} - \frac{\sin(2k_0 d \tan \beta)}{2k_0 d \tan \beta} \right\}, \quad (7.8)$$

so that

$$m(\mathscr{C}_b) = \tfrac{1}{2} \sec^2 \beta \left\{ 1 - \frac{\sin(D \tan \beta)}{\tan \beta \sinh D} \right\}, \quad D = 2k_0 d. \quad (7.9)$$

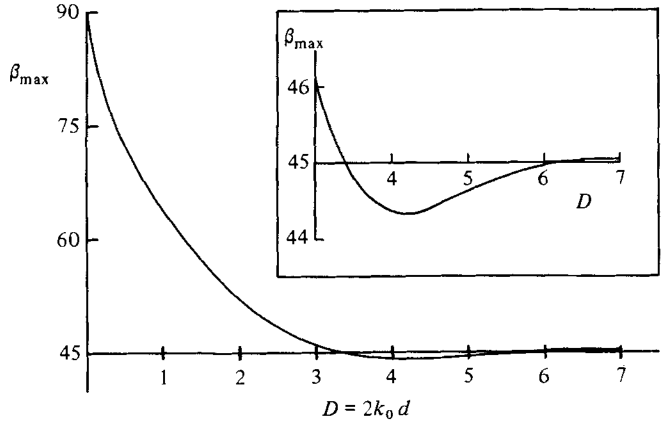

FIGURE 7. The value of β_{max}, computed from (7.13), plotted against the dimensionless depth $D = 2k_0 d$, where $k_0 \tanh(k_0 d) = \omega^2/g$.

Thus the maximum angle β_{max} (where $m(\mathscr{C}_b) = 1$) is given by

$$\frac{2}{1+t^2} = 1 - \frac{\sin(Dt)}{t \sinh D}, \quad t = \tan \beta_{max}. \tag{7.10}$$

This can easily be solved numerically by iteration, starting from $t = t_0$, where

$$t_0 = \begin{cases} (12D^{-2})^{\frac{1}{4}} & (D < D_0 = 2.5, \text{ say}), \\ 1 + \dfrac{\sin D}{\sinh D} & (D \geqslant D_0), \end{cases}$$

these being the asymptotic forms in the respective limits as $D \to 0$ and $D \to \infty$.

However, a small improvement on this value of β_{max} can be made by utilizing the imaginary part of (7.4), viz

$$0 = \int_{\mathscr{C}_b} \left\{ \cosh k_0(d-y) \cos k_0(x-b) \frac{\partial \phi}{\partial n} + \sinh k_0(d-y) \sin k_0(x-b) \frac{\partial \phi}{\partial s} \right\} ds. \tag{7.11}$$

An arbitrary multiple γ times this equation is now added to (7.5), and the bound for $K\phi^2(b, 0)$ is constructed as in (7.6)–(7.8), to give, in place of (7.10),

$$\left. \begin{aligned} \frac{2}{1+t^2} &= (1-A) + 2\gamma B + \gamma^2(1+A), \\ \text{where} \qquad t \sinh D(A, B) &= (\sin(Dt), 1 - \cos(Dt)). \end{aligned} \right\} \tag{7.12}$$

We desire β_{max} to be as large as possible, so we minimize this quadratic in γ by choosing $\gamma = -B/(1+A)$ to give

$$\frac{2}{1+t^2} = 1 - A - \frac{B^2}{1+A} = \frac{t \sinh D - 2B}{t \sinh D + \sin(Dt)}. \tag{7.13}$$

Once this is solved by iteration, to give β_{max} as shown in figure 7, then straight lines with $0 \leqslant \beta \leqslant \beta_{max}$ can be used in the uniqueness proof. The main improvement of (7.13) over (7.10) is seen for small D, where now $t \sim (48D^{-2})^{\frac{1}{4}}$. β_{max} is, surprisingly, not monotonic, but crosses the value 45° when $D = 2n\pi$ and when

$$\sinh D + 2 \tan(\tfrac{1}{2}D) = 0. \tag{7.14}$$

the lowest root of which is 3.40578.... In fact $\beta_{max} \geqslant 44\frac{1}{3}°$ for all values of D, that is, for all frequencies and depths.

Although great attention has been paid to β_{max} obtained by this method, it must be emphasized that β_{max} (and its infinite-depth value, $\frac{1}{4}\pi$) should not be seen as intrinsic to the uniqueness problem; the value is purely a result of the method used to obtain the bound for $K\phi^2(b,0)$. It may well be possible to achieve higher angles of slope for straight lines by constructing the bound in a different way and possibly by considering other functions ψ. It is certainly possible to make use of other functions $x(y)$ in (7.7), just as in §6, and the curves can be chosen to fit the particular geometry under consideration.

8. Conclusion

A method has been developed which demonstrates uniqueness in a large class of two-dimensional linear water-wave problems. The techniques can be seen as a generalization of work by John (1950), and derives bounds for $\phi^2(x,0)$ by considering integrals along non-vertical lines. In infinite depth, when these lines are straight, they can be inclined up to an angle 45° away from the vertical; for lines that are not straight the possible inclinations are greater. This technique also proves that the potential will be unique in almost every case (with a very limited class of exception) provided the frequency is small enough; similarly, we show that submerged bodies give rise to unique potentials provided the frequency is large enough. In either limit, the method gives a bound on how small, or large, the frequency must be for uniqueness to be guaranteed.

Finally the technique is used in the finite-depth case, where the maximum angle of slope, of straight lines to be used in the proof, is a function of the dimensionless depth, and is given by an implicit relationship. This is solved numerically, and it is shown that the maximum angle is at least $44\frac{1}{3}°$ for all depths of fluid.

Further work will attempt to extend the method herein to three-dimensional situations.

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